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VIBRATIONS OF THE SURFACE OF AN ELASTIC DOUBLE-LAYERED HALF-SPACE WITH A PERIODIC SYSTEM OF CRACKS†

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The two-dimensional problem of the normal incidence of a plane transverse wave from the far field on to the free surface of an elastic double-layered half-space, comprising a homogeneous layer attached to a semi-infinite base of a different elastic material, is considered. At the boundary between the two media there is a system of plane cracks, arranged periodically along the separation line, which models the fracture zone at the interface between dense solid rock and soft sedimentary rock. The effect of the fractures on the transmission of a transverse seismic wave generated by a deep-focus earthquake, and of the type of vibrations of the free surface of the ground that result, is studied. It is difficult to predict whether the seismic wave is strengthened or weakened by the fracture zone. The effect of the system of cracks on vibrations of the free surface largely depends on the physical and geometrical parameters and, primarily, on the vibration frequencies. © 1998 Elsevier Science Ltd. All rights reserved.

1. Suppose a plane transverse wave is incident at right-angles on a periodic system of plane cracks situated along the boundary between an elastic layer of thickness d and an elastic half-space, of different homogeneous isotropic materials (Fig. 1). This is a model of the transmission of a seismic wave coming from a lower half-space (medium 1) and interacting with a fracture zone at its boundary with sedimentary rock (medium 2). The distance between two adjacent cracks is equal to $2b$ and the lattice period is $2a$ ($a > b$). We shall investigate the problem in a two-dimensional formulation.

Using a Laplace transformation, we can reduce the unsteady problem to a steady problem with a time dependence in the form $\exp(-i\omega t)$ for each frequency ω of the real spectrum of a seismic wave.

In both the lower and upper media we use a Lamé representation for the displacement vector [1] (assuming the deformation to be two-dimensional)

$$u_x = \partial\varphi/\partial x + \partial\psi/\partial y,\ \ u_y = \partial\varphi/\partial y - \partial\psi/\partial x,\ \ u_z = 0 \tag{1.1}$$

The stresses can be expressed in terms of the potentials of longitudinal (φ) and transverse (ψ) waves using the generalized Hooke's law. As we know, both potentials satisfy the Helmholtz equation

$$\Delta\varphi + k_p^2\varphi = 0,\ \ \Delta\psi + k_s^2\psi = 0 \tag{1.2}$$

The boundary conditions have the form

$$x = d : \sigma_{xx2} = 0,\ \ \tau_{xy2} = 0;\ -\infty < y < \infty \tag{1.3}$$

$$x = 0 : \sigma_{xx1} = \sigma_{xx2} = 0,\ \ \tau_{xy1} = \tau_{xy2} = 0;\ \ b < |y| < a \tag{1.4}$$

$$x = 0 : \sigma_{xx1} = \sigma_{xx2},\ \ \tau_{xy1} = \tau_{xy2},\ \ u_{x1} = u_{x2},\ \ u_{y1} = u_{y2};\ \ |y| < b \tag{1.5}$$

where we have allowed for the fact that, by virtue of the natural periodicity of the problem with respect to the y coordinate, it is sufficient to consider the strip $|y| < a$ alone.

Taking the same periodicity into account, we have the following representations for the potentials in the first (lower) and second (upper) media [2] (everywhere below the summation is taken over n from 1 to ∞)

$$\varphi_1 = \Sigma A_n s_n \exp(q_{1n}x),\ \ \psi_1 = \psi_0 \exp(ik_{s1}x) +$$
$$+R\exp(-ik_{s1}x) + \Sigma B_n c_n \exp(r_{1n}x),\ \ -\infty < x < 0 \tag{1.6}$$

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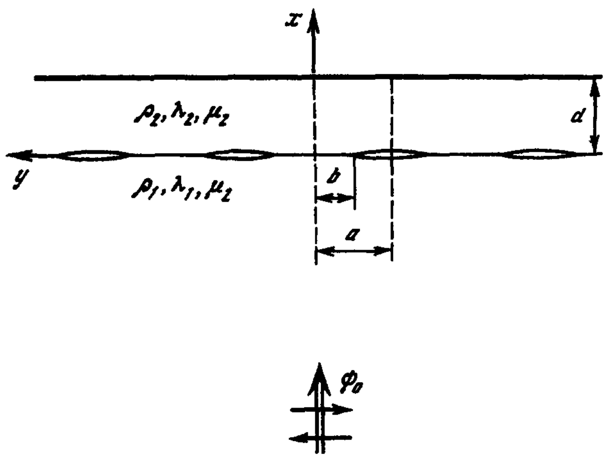

Fig. 1.

$$\varphi_2 = \Sigma\left[C_n \operatorname{ch} q_{2n}(x-d) + D_n \frac{\gamma_{2n}}{\alpha_{2n}} \operatorname{sh} q_{2n}(x-d)\right] s_n$$

$$\psi_2 = W \sin k_{s2}(x-d) + \Sigma\left[D_n \operatorname{ch} r_{2n}(x-d) + C_n \frac{\gamma_{2n}}{\beta_{2n}} \operatorname{sh} r_{2n}(x-d)\right] c_n, \quad 0 < x < d \tag{1.7}$$

Here

$$s_n = \sin \Pi_n y, \quad c_n = \cos \Pi_n y, \quad \Pi_n = \pi n / a$$

$$q_{jn} = (\Pi_n^2 - k_{pj}^2)^{1/2}, \quad r_{jn} = (\Pi_n^2 - k_{sj}^2)^{1/2} \tag{1.8}$$

$$\alpha_{jn} = 2\Pi_n q_{jn}, \quad \beta_{jn} = 2\Pi_n r_{jn}, \quad \gamma_{jn} = 2\Pi_n^2 - k_{sj}^2, \quad j = 1, 2$$

where k_{p1} and k_{p2}, k_{s2} are the wave numbers for longitudinal and transverse waves in the first and second media, respectively.

Note that expressions (1.7) automatically satisfy the boundary condition on the free surface (1.3).

2. We shall need expressions for the displacements and stresses at the interface of the two media (at $x = 0$)

$$u_{x1} = \Sigma(q_{1n} A_n - \Pi_n B_n) s_n$$

$$u_{y1} = ik_{s1}(R - \psi_0) + \Sigma(\Pi_n A_n - r_{1n} B_n) c_n \tag{2.1}$$

$$\sigma_{xx1} / \rho_1 = c_{s1}^2 \Sigma(\gamma_{1n} A_n - \beta_{1n} B_n) s_n$$

$$\tau_{xy1} / \rho_1 = \omega^2(\psi_0 + R) + c_{s1}^2 \Sigma(\alpha_{1n} A_n - \gamma_{1n} B_n) c_n \tag{2.2}$$

$$u_{x2} = \Sigma(-\delta_n C_n + \zeta_n D_n) s_n$$

$$u_{y2} = -k_{s2} W \cos k_{s2} d + \Sigma(-\gamma_n C_n + \mu_n D_n) c_n$$

$$\sigma_{xx2} / \rho_2 = c_{s2}^2 \Sigma[\varepsilon_n C_n - D_n \xi_n / (2\Pi_n q_{2n})] s_n$$

$$\tau_{xy2} / \rho_2 = -\omega^2 W \sin k_{s2} d + c_{s2}^2 \Sigma[C_n \eta_n / (2\Pi_n r_{2n}) + \varepsilon_n D_n] c_n \tag{2.4}$$

Here

$$\delta_n = q_{2n} \operatorname{sh} q_{2n} d - \frac{\gamma_{2n}}{2r_{2n}} \operatorname{sh} r_{2n} d$$

$$\varepsilon_n = \gamma_{2n}(\operatorname{ch} q_{2n} d - \operatorname{ch} r_{2n} d), \quad \zeta_n = \frac{\gamma_{2n}}{2\Pi_n} \operatorname{ch} q_{2n} d - \Pi_n \operatorname{ch} r_{2n} d$$

$$\xi_n = \gamma_{2n}^2 \operatorname{sh} q_{2n} d - \alpha_{2n}\beta_{2n} \operatorname{sh} r_{2n} d, \quad \eta_n = \gamma_{2n}^2 \operatorname{sh} r_{2n} d - \alpha_{2n}\beta_{2n} \operatorname{sh} q_{2n} d \qquad (2.5)$$

$$\mu_n = r_{2n} \operatorname{sh} r_{2n} d - \frac{\gamma_{2n}}{2q_{2n}} \operatorname{sh} q_{2n} d, \quad \nu_n = \frac{\gamma_{2n}}{2\Pi_n} \operatorname{ch} r_{2n} d - \Pi_n \operatorname{ch} q_{2n} d$$

To determine the constants A_n, B_n, C_n, D_n, we will introduce two unknown functions $g^\sigma(y)$ and $g^\tau(y)$, associated with the normal and shear stress in the gap (1.5) by

$$\sigma_{xx1} = \sigma_{xx2} = \begin{cases} g^\sigma(y), & |y| < b \\ 0, & b < |y| < a \end{cases}, \quad x = 0$$

$$\tau_{xy1} = \tau_{xy2} = \begin{cases} g^\tau(y), & |y| < b \\ 0, & b < |y| < a \end{cases}, \quad x = 0 \qquad (2.6)$$

Then, using the orthogonality of the trigonometric functions and the continuity of the stresses (1.5) on crossing the interface between the two media, we can express all the unknown coefficients in terms of the above functions (everywhere below the integration is performed over the segment $[-b, b]$)

$$A_n = E_{1n}(\gamma_{1n} G_n^\sigma - \beta_{1n} G_n^\tau), \quad B_n = E_{1n}(\alpha_{1n} G_n^\sigma - \gamma_{1n} G_n^\tau)$$

$$C_n = E_{2n}\beta_{2n}(\alpha_{2n}\varepsilon_n G_n^\sigma - \xi_n G_n^\tau), \quad D_n = E_{2n}\alpha_{2n}(-\eta_{2n} G_n^\sigma + \beta_{2n}\varepsilon_n G_n^\tau)$$

$$E_{jn} = (a\rho_j c_{sj}^2 \Delta_{jn})^{-1}, \quad j = 1, 2 \qquad (2.7)$$

$$\Delta_{1n} = \gamma_{1n}^2 - \alpha_{1n}\beta_{1n}, \quad \Delta_{2n} = \alpha_{2n}\beta_{2n}\varepsilon_n^2 + \xi_n\eta_n$$

$$G_n^\sigma = \int g^\sigma(\eta) s_n d\eta, \quad G_n^\tau = \int g^\tau(\eta) c_n d\eta$$

(the quantity Δ_{1n} is associated with the Rayleigh function).

Then, in the light of the other continuity conditions for displacements (1.5), we obtain a system of two integral equations of the first kind with different kernels relative to the functions $g^\sigma(y)$ and $g^\tau(y)$

$$\int K_{j1}(y-\eta) g^\tau(\eta) d\eta + \int K_{j2}(y-\eta) g^\sigma(\eta) d\eta = L_j, \quad j = 1, 2; \quad |y| < b$$

$$L_1 = 2ik_{s1}\psi_0 - DG^\tau, \quad L_2 = 0 \qquad (2.8)$$

where

$$G^\tau = \int g^\tau(\eta) d\eta, \quad D = \frac{ik_{s1}}{2a\rho_1\omega^2} - \frac{k_{s2}}{2a\rho_2\omega^2} \operatorname{ctg} k_{s2} d \qquad (2.9)$$

$$K_{11} = -\Sigma\left[E_{1n}k_{s1}^2 r_{1n} - E_{2n}\beta_{2n}(\nu_n\xi_n - \mu_n\alpha_{2n}\varepsilon_n)\right]c_n$$

$$K_{12} = \Sigma\left[E_{1n}\Pi_n(2q_{1n}r_{1n} - \gamma_{1n}) + E_{2n}\alpha_{2n}(\nu_n\beta_{2n}\varepsilon_n + \mu_n\eta_n)\right]s_n$$

$$K_{21} = -\Sigma\left[E_{1n}\Pi_n(2q_{1n}r_{1n} - \gamma_{1n}) + E_{2n}\beta_{2n}(\zeta_n\alpha_{2n}\varepsilon_n - \delta_n\xi_n)\right]s_n \qquad (2.10)$$

$$K_{22} = -\Sigma\left[E_{1n}k_{s1}^2 q_{1n} - E_{2n}\alpha_{2n}(\delta_n\beta_{2n}\varepsilon_n + \zeta_n\eta_n)\right]c_n$$

3. We will now consider a system of integral equations differing from (2.8) in the fact that $L_1 = 1$ (to be called the auxiliary system below). We have

$$g^{\nu}(y) = (2ik_{s1}\psi_0 - DG^{\tau})h^{\nu}(y), \quad \nu = \tau, \sigma \tag{3.1}$$

Hence we obtain the following relation between the integral characteristics

$$G^{\tau} = 2ik_{s1}\psi_0 H^{\tau}/(1 + DH^{\tau}), \quad H^{\tau} = \int h^{\tau}(\eta)d\eta \tag{3.2}$$

We now introduce a transmission of function, equal to the ratio of the amplitude of fibration of the free surface $u_y(x = d, y)$ to the oscillation amplitude $u_0 = -ik_{s1}\psi_0$ in the incident wave

$$F(\omega, y) = -\frac{2}{1 + DH^{\tau}}\left[\frac{H^{\tau}}{2a\omega\rho_2 C_{s2}\sin k_{s2}d} + k_{s2}^2 \Sigma E_{2n} r_{2n}(\alpha_{2n}\gamma_{2n}\varepsilon_n H_n^{\sigma} + \xi_n H_n^{\tau})c_n\right]$$

$$H_n^{\sigma} = \int h^{\sigma}(\eta)s_n d\eta, \quad H_n^{\tau} = \int h^{\tau}(\eta)c_n d\eta \tag{3.3}$$

Figure 2 shows an example of a direct numerical calculation of the modulus of the transmission function $F(\omega, 0)$ calculated for certain real parameters of the primary and sedimentary layers at $y = 0$ ($\rho_1 = 2.2$ g/cm, $c_{p1} = 2.80$ km/s, $c_{s1} = 1.10$ km/s, $\rho_2 = 2.0$ g/cm, $c_{p2} = 0.36$ km/s, $c_{s2} = 0.21$ km/s). We can compare the numerical solution for $d/a = 1$, $b/a = 0.5$ (the dashed line) with the classical solution [3] of the one-dimensional problem without cracks (the solid line, $b/a = 1$). The latter is easily obtained as a special case using the above method. In fact, if $b = a$, by virtue of the orthogonality of the trigonometric functions in the representations of the kernels (2.10), the solution of system (2.8) has the form

$$g^{\tau}(y) \equiv g^{\tau} = ik_{s1}\psi_0/(bD), \quad g^{\sigma}(y) \equiv 0 \tag{3.4}$$

Then $H_n^{\tau} = H_n^{\sigma} = 0$ ($n = 1, 2, \ldots$). Moreover, it follows from Eq. (3.1) for $\nu = \tau$ that

$$H^{\tau} = \frac{G^{\tau}}{2ik_{s1}\psi_0 - DG^{\tau}} \to \infty \quad \text{as} \quad b \to a \tag{3.5}$$

since the denominator of the last fraction tends to zero, by virtue of Eq. (3.4). Thus expression (3.3) takes the form

$$F(\omega, y) = 2\left(\cos k_{s2}d - i\frac{\rho_{s2}c_{s2}}{\rho_{s1}c_{s1}}\sin k_{s2}d\right)^{-1} \tag{3.6}$$

which is identical with the classical result of seismology [3].

Obviously, in frequency ranges for which the dashed curve in Fig. 2 lies above the solid curve, the presence of a fracture zone increases the intensity of vibration of the free surface. For those frequencies where the dashed curve is below the solid curve, the cracks protect the surface from an incident seismic wave. For a given fixed frequency it is obviously impossible to predict which of these two cases applies: the result largely depends on the relation between the physical and geometrical parameters. However, one can draw the general conclusion that

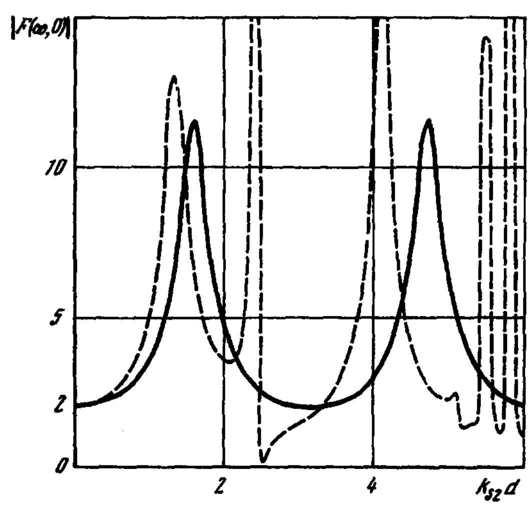

Fig. 2.

fracture zones could be very hazardous for installations situated above them, since there are certain frequencies of the earthquake spectrum for which the vibration amplitude has local maxima of high intensity.

The auxiliary system was solved numerically by the collocation method. It can be shown that the kernels $K_{11}(y)$ and $K_{22}(y)$ have a logarithmic singularity as $y \to 0$. The kernels $K_{12}(y)$ and $K_{21}(y)$ behave as follows: $O(|y|)$, $y \to 0$. These properties of the kernels ensure that the collocation method yields a stable solution of the auxiliary system.

4. If the two media are of the same material (the system of cracks lies in a homogeneous medium), an explicit analytic solution can be constructed in one special case. Assuming that: (a) the upper layer is quite thick: $d/a \gg 1$, (b) the vibration frequency is low: $k_{s2}d \ll 1$, we have

$$K_{11}(y) \approx K_{22}(y) \approx a_{11}\Sigma\frac{c_n}{n} = -a_{11}\ln\left|2\sin\frac{\pi y}{2a}\right|, \quad K_{12}(y) \approx K_{21}(y) \approx 0$$

$$a_{11} = \frac{1}{2\pi}\left[\frac{1}{\rho_1 c_{s1}^2(1-\delta_1^2)} + \frac{1}{\rho_2 c_{s2}^2(1-\delta_2^2)}\right], \quad \delta_j = \frac{k_{pj}}{k_{sj}}, \quad j = 1,2$$

It follows that the normal stress component $h^{\sigma}(y)$ in the auxiliary system is zero, and the second unknown function is found from an integral equation, the solution of which has the form [4]

$$h^{\tau}(y) = -\eta\left(2aa_{11}\sqrt{\xi_0^2 - \xi^2}\ln\xi_0\right)^{-1}$$

$$\xi = \sin\frac{\pi y}{2a}, \quad \xi_0 = \sin\frac{\pi b}{2a}, \quad \eta = \cos\frac{\pi y}{2a} \tag{4.1}$$

In addition, under assumptions (a) and (b), expression (3.3) for the transmission function is independent of y and takes the form

$$F(\omega) = 2H^{\tau}\left[H^{\tau}\exp(-ik_{s2}d) - 2a\omega\rho_2 c_{s2}\sin k_{s2}d\right]^{-1}$$
$$H^{\tau} = \int h^{\tau}(\eta)d\eta = -(a_{11}\ln\xi_0)^{-1} \tag{4.2}$$

This is also an explicit approximate analytic representation for the transmission function for a homogeneous medium with surface cracks.

It turns out that approximation (4.2) is surprisingly accurate for all (not only small) $k_{s2}d$, if $d/a \geq 1$. Figure 3 shows the example $\rho_1 = \rho_2 = 2.2$ g/cm, $c_{p1} = c_{p2} = 1.8$ km/s, $c_{s1} = c_{s2} = 1.1$ k/s, $d/a = 2.5$, and $b/a = 0.5$, where

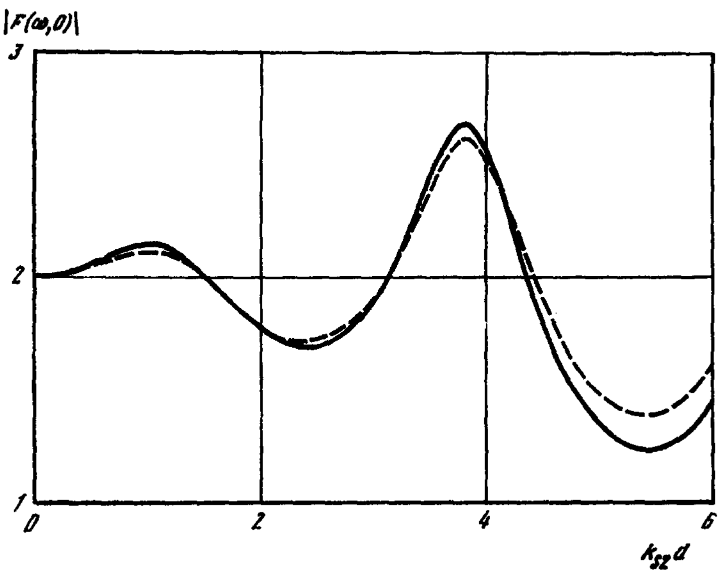

Fig. 3.

the exact numerical solution (the solid curve) is compared with the analytic solution (the dashed curve). For $d/a \geqslant 3$, the analytic approximation is accurate to within 5% uniformly with respect to $k_{s2}d \in (0, 2\pi]$.

We can draw some conclusions from the explicit approximation (4.1).

1. If $k_{s2}d = \pi m$ $(m = 0, 1, \ldots)$ we have $|F(\omega)| = 2$, that is, for these fixed frequencies the amplitude of the vibration of the free surface is the same as in the one-dimensional problem without cracks, whatever the value of the relative gap b/a.

2. As the gap between cracks $b/a \to 0$, we have $\ln \xi_0 \to -\infty$. Hence $H^\tau \to 0$. Thus, $|F(\omega)|$ tends to zero beyond the neighbourhood of the above frequencies. However, the properties of the logarithmic function ensure that this is a very slow process. For practical values of the geometrical parameters, it is therefore unlikely that the existence of a periodic system of cracks would be able to protect the boundary from an incoming seismic wave.

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